Math 275D Lecture 25 Notes

Daniel Raban

December 4, 2019

1 Dyson Brownian Motion

1.1 The drifting term in the rate of change of the eigenvalues

Let's say we have Hermitian matrices $H^t = H^0 + H_G^t$, where $H^t = (H^t)^\dagger$ and $H^0 = (H^0)^\dagger$. Let λ_k^r be the k-th largest eigenvalue of H^t and u_k^t be such that $\|U_k^t\|_2 = 1$ and $H^t u_k^t = \lambda_k^t u_k^t$ (eigenvector).

We let $[H_G^t]_{i,j} \sim B(t)$ for $i < j$ and $[H_G^t]_{i,i} \sim B(2t)$. The entries of H^t are independent. with $H_G^t = (H_G^t)^\dagger$. Then $H_{i,j}^t = H_{i,j}^0 + B_{i,j}(t)$ is a process for each $i \leq j$. So we can consider eigenvalues as a function of these processes: $\lambda_k^t(\{H_{i,j}\}_{i \leq j})$. What if we change the eigenvalues a little bit?

$$\begin{aligned} d\lambda_k^t &= \sum_{i,j} \frac{\partial \lambda_k}{\partial H_{i,j}^t} \, dB_{i,j}^t + \frac{1}{2} \sum_{i,j,i',j'} \frac{\partial^2 \lambda_k}{\partial H_{i,j} \partial H_{i',j'}} \, dB_{i,j} \, dB_{i',j'} \\ &= \nabla \lambda_k \cdot dH^t + dH^t \cdot \mathrm{Hess}(\lambda_k) \, dH^t. \end{aligned}$$

Since these $N \times N$ matrices are Hermitian, we have N^2 processes which are not all independent.

Since $dB_{i,j} dB_{i',j'} = dt$ if $\{i,j\} = \{i',j'\}$ and 0 otherwise, we get

$$d\lambda_k^t = \sum_{i,j} \frac{\partial \lambda_k}{\partial H_{i,j}^t} \, dB_{i,j}^t + \frac{1}{2} \sum_{i,j} \left[\frac{\partial \lambda_k}{\partial H_{i,j}^2} + \frac{\partial^2 \lambda_k}{\partial H_{i,j} \partial H_{j,i}} \right] dt$$

There are results that say

$$\frac{\partial \lambda_k}{\partial_{H_{i,j}}} = u_k(i) u_k(j), \qquad \frac{\partial u_k}{\partial H_{i,j}} = \sum_{\ell \neq k} \frac{u_k(i) u_\ell(j)}{\lambda_\ell - \lambda_k} u_\ell.$$

So the right term, the **"drifting term"**, is

$$\sum_\ell \frac{-1}{\lambda_\ell - \lambda_k} \, dt.$$

So if λ_k is very close to λ_{k+1}, we get a very large negative term. So there is some large force to pull λ_k down, away from λ_{k+1}. This gives us a surprising property: the order of the eigenvalues never changes! That is, if we plot all the eigenvalues with t, the curves never intersect.

1.2 The diffusion term

Let's do some calculations with the first term, the **"diffusion term"**. This is

$$\sum_{i,j} u_k(i)u_k(j)\,dB_{i,j}(t)$$

Now define

$$B_k(t) = \int \sum_{i,j} u_k^t(i)u_k(j)\,dB_{i,j}(t).$$

This is a process, and we can check that for any fixed t, $B_k(t) \sim \mathcal{N}(0,t)$: We can calculate

$$\mathbb{E}[B_k^2(t)] = \sum_{i,j} u_k^t(i)u_k^t(j)u_k^t(i)u_k^t(j) \cdot t = t.$$

We can also calculate for $\ell \neq k$,

$$\mathbb{E}[B_k(t)B_\ell(t)] = \sum_{i,j} i_k^t(i)u_k^t(j)u_\ell(i)u_\ell(j) = 0$$

because the u_k, u_ℓ are orthogonal to each other. In fact, we can prove that

$$\mathbb{E}[B_k^m(t)] = \mathbb{E}[\mathcal{N}(0,t)^m].$$

Moreover, $B_k(\cdot)$ is a Brownian motion, and $\{B_k\}_{k=1}^N$ are independent.

So we have that

$$d\lambda_k = dB_k + \sum_\ell \frac{-1}{\lambda_\ell - \lambda_k}.$$

What if I define some other process

$$d\widetilde{\lambda}_k = d\widetilde{B}_k + \sum_\ell \frac{-1}{\widetilde{\lambda}_\ell - \widetilde{\lambda}_k}$$

with the same distribution? This doesn't come from a matrix, but we can still use it to study the distribution of λ_k.